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A Class Of Social Choice Mechanisms Possessing Pure-Strategy Nash Equilbria

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Abstract
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A CLASS OF SOCIAL CHOICE MECHANISMS

POSSESSING PURE-STRATEGY NASH EQUILIBRIA

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ABSTRACT

A class of social choice mechanisms is described. Each mechanism in the class, when viewed as a noncooperative game, is shown to possess at least one Nash equilibrium in pure strategies. Some members of the class are useful in social choice applications.

1. Description

Imagine a society consisting of n agents ($i=1, \ldots, n$) who are to choose among m social states ($j=1, \ldots, m$); n is at least 2, m is at least 3. Each agent i is assumed to have a complete, reflexive, and transitive ordering $R(i)$ of the social states. Let r_{ij} represent the number of social states which agent i strictly prefers to social state j; then

$$(1) \quad j \begin{Bmatrix} R(i) \\ P(i) \\ I(i) \end{Bmatrix} j' \text{ iff } \qquad r_{ij} \begin{Bmatrix} \leq \\ < \\ = \end{Bmatrix} r_{ij'}$$

There is an initial endowment of rights, $w(i)$, an integer-valued function satisfying

$$(2) \quad w(i) \geq 0 \quad \text{for all } i$$

$$\sum_i w(i) = m - 1.$$

An agent has the right to object to individual social states. Write $x_{ij} = 1$ if agent i objects to social state j, $x_{ij} = 0$ otherwise. Every player makes the maximum number of objections.

$$(3) \quad \sum_j x_{ij} = w(i) \qquad \text{for all } i.$$

$$x_{ij} = 0 \text{ or } 1 \quad \text{for all } i \text{ and } j.$$

A social state j is not chosen if some player objects to it; thus, social state j is chosen if and only if

(4) $\sum_i x_{ij} = 0$ Clearly, at least one social state is chosen.

Thus, the social choice mechanism finds for the society [(i), (j), (R(i)), w(i)] a social choice based on the revealed objections x_{ij}.

2. Pure-Strategy Nash Equilibria

When one of the above mechanisms is viewed as a noncooperative game in normal form, each player i has for a strategy space the set of vectors $x_i = (x_{i1}, \ldots, x_{im})$ satisfying (3). In this context we prove the following:

<u>Theorem</u>: All mechanisms satisfying (2) - (4) possess at least one pure-strategy Nash equilibrium.

<u>Proof</u>: Let $I = \{i: w(i) > 0\}$, relabeled if necessary so that $I = \{1, \ldots, k\}$. where $k \leq n$.

Let $1 \in I$ maximize $\sum_j r_{1j} x_{1j}$

$$\text{S.T.} \quad \sum x_{1j} = w(1)$$
$$x_{1j} = 0 \text{ or } 1.$$

It is clear that a solution exists; call it $x_1{}^*$.

Now $2 \in I$ maximizes $\sum_j r_{2j} x_{2j}$

$$\text{S.T.} \quad \sum x_{2j} = w(2)$$
$$x_{2j} = 0 \text{ or } 1$$
$$\text{and} \quad \sum_j x_{2j}\, x^*_{1j} = 0, \text{ at } x^*_2.$$

The maximization sequence continues until $k \in I$ maximizes

$$\sum_j r_{kj} x_{kj}$$

$$\text{S.T.} \quad \sum x_{kj} = w(k)$$

$$x_{ki} = 0 \text{ or } 1.$$

$$\text{and} \quad \sum_j x_{kj} \left(\prod_{i=1}^{k-1} x^*_{ij} \right) = 0$$

at x^*_k.

The claim then is that $x^* = (x^*_1, \ldots, x^*_k)$ is a Nash equilibrium.

By the construction there exists an unique social state h such that $\Sigma x^*_{ih} = 0$, which social choice corresponds to the Nash equilibrium. Suppose there exists $i \in I$ and social state h' such that $r_{ih'} < r_{ih}$ and $x^*_{ih'} = 1$; that is, i has objected to h' which he happens to prefer to the social choice h. Such must surely happen if x^* is not a Nash equilibrium. Now the strategy $x_{ij} = x^*_{ij}$ for $j = h, h'$

$x_{ih} = 1$

$x_{ih'} = 0$

is feasible but

$$\sum_j r_{ij} (x_{ij} - x^*_{ij}) > 0$$

contradicting the definition of x^*_i. Since i was chosen arbitrarily, x^* is indeed the desired Nash equilibrium.

Social state j is a weak Pareto optimum if there is no social state j' such that for all individuals i, $jP(i)j'$.

<u>Corollary</u>: If x^* solves the above maximization sequence and j is the chosen social state, then j is a weak Pareto optimum.

<u>Proof</u>: If for all i, $j'P(i)j$, then the player who objected to j' does better by objecting to j instead, contradicting that x^* is a Nash equilibrium.

This assertion cannot be strengthened to strong Pareto optimum, as the following example shows.

<u>Example 1</u>: $n = 2$, $m = 3$, $w(i) = 1$ for both i. Preferences are $1P(1)2P(1)3$, $3P(2)1I(2)2$. There are two pure-strategy Nash equilibria, $((0,0,1), (0,1,0))$ and $((0,0,1), (1,0,0))$. The social state corresponding to the latter, social state 2, is not a strong Pareto optimum.

In the case where the preferences of all agents are strict orders, the social state corresponding to the constructed Nash equilibrium will however be a strong Pareto optimum.

Since the labeling of the members of I in the proof is arbitrary, there are in principle at most $k!$ Nash equilibria so constructible. However, not all the Nash equilibria are solutions of such a maximization sequence. Witness the following example.

<u>Example 2</u>: Like example 1, only now preferences are $1I(1)2P(1)3$, $1I(2)2P(2)3$. Then $((0,0,1), (0,0,1))$ is a Nash equilibrium, although not constructible by the method of the theorem.

For the situation of this example to obtain, all agents must be indifferent among the set of social states not objected to.

The equilibrium set consists of all social states j such that for each j there exists a Nash equilibrium which leads to the choice of j. By the above discussion, any member of the equilibrium set is a weak Pareto optimum. The idea of the equilibrium set is that it is not the strategic maneuverings so much as the chosen social state, that is of interest in social choice theory.

3. Applications

The class of social choice mechanisms described above is useful for applications in social choice theory. Three such applications are made here: to Gibbard's theorem on straightforward game forms [1], to Sen's Paradox of the Paretian liberal [5,6], and to Rawls' theory of justice [4].

A game form is a function from the product space of individual strategies to the space of social states. In a game form, individual utilities have not yet been attached to outcomes (social states). A game form is straightforward for player i, if given R(i), he has a dominant strategy. A player is a dictator for a game form, if, for every social state j, he has a strategy, such that j is the social state chosen. Dictatorial mechanisms of the present class require that $w(k) = m - 1$ for some i. Gibbard has shown that every straightforward game form with at least three possible outcomes is dictatorial. Clearly, a straightforward game form possesses at least one pure-strategy Nash equilibrium, each player playing a dominant strategy. The class of mechanisms satisfying (2) - (4) shows that straightforwardness is not a necessary condition for possession of pure strategy Nash equilibria, once utilities are attached to outcomes. Indeed, the situation may be straightforward for no player. Take the case 1P(1)2P(1)3, 2P(2)1P(2)3, each player having one objection. Both ((0,1,0), (0,0,1)) and ((0,0,1), (1,0,0)) are Nash equilibria, but neither player has a dominant strategy. Nor is it at all clear in such a situation (as Gibbard himself points out [1, p. 589] what counts as manipulation on the part of an individual player. If what one is after are Nash equilibria, it is possible to escape the "regrettable consequences" of Gibbard's result on game forms.

Sen's Liberal Paradox [5, pp. 78-88] shows that on a certain view of liberalism, liberal values conflict with the weak Pareto principle. This result has spurred considerable research activity (the bibliography in [6] contains over fifty items). An argument can be made for the following expression of liberal principle: every individual has the right to object to at least one social state. In order for this claim to be logically consistent in the present framework, one requires the following strengthening of (2):

$$(2)' \qquad w(k) > 0 \text{ for all } i$$
$$\Sigma w(i) = m - 1 \geq n.$$

As seen above, the equilibrium set of such mechanisms does consist of Pareto optima; as such, they invite comparison with the mechanisms suggested by Gibbard [2, Thm. 4]. On the one hand, mechanisms satisfying (2)' dispense with Gibbard's notion of waiving rights (for the difficulties of which see [3]); on the other hand, they relax Gibbard's structural requirement that $m \geq 2^n$. Thus, mechanisms satisfying (2)' appear satisfactory on formal grounds. Again, in the paradigm cases of "Lady Chatterly's Lover" [5], "Edwin vs. Angelina" [2], and "Work Choice" [6], mechanisms satisfying (2)' yield reasonable outcomes on intuitive liberal grounds. If rights are something a person can stand on against the rest of society, then mechanisms satisfying (2)' express such rights.

As a final application, consider the following problem from the theory of justice. Suppose a social state is a distribution of utility, which is comparable and additive but not transferable. There are n + 1 social states, the first n of which constitute the standard basis for Euclidean n-space. The last social state m is given by

$$m = (u_1, \ldots, u_n)$$
$$0 < u_i < 1$$
$$\Sigma u_i = 1.$$

In negotiations before a veil of ignorence, each agent is allowed one objection. All Nash equilibria correspond to social state m. Now if one interprets the first n social states as expressing the various egoistic conceptions of justice, it is clear that egoistic conceptions are unacceptable before a veil of ignorance. Further, not even presenting egoistic conceptions in an Original Position (for instance, in Rawls' theory of justice [4, p. 136]) is justified, since principles unacceptable before a veil of ignorance remain unacceptable behind a veil of ignorance.

REFERENCES

1. Gibbard, A. "Manipulation of Voting Schemes: A General Result," Econometrica. 41 (1973), 587-601.

2. __________. "A Pareto-Consistent Libertarian Claim," Journal of Economic Theory. 7 (1974), 388-410.

3. Kelley, J. S. "Rights Exercising and a Pareto-Consistent Libertarian Claim," Journal of Economic Theory 13 (1976), 138-153.

4. Rawls, J. A Theory of Justice. Cambridge, Mass.: Harvard University Press, 1971.

5. Sen, A. K. Collective Choice and Social Welfare. San Francisco: Holdren-Day, 1970.

6. __________. "Liberty, Unanimity and Rights." Economica 43 (1976), 217-245.